Activity: Fold Four Boxes

1. Cut out your copy of the crease pattern for the square-base twist box—but only cut along the solid lines.

2. Look at this key:

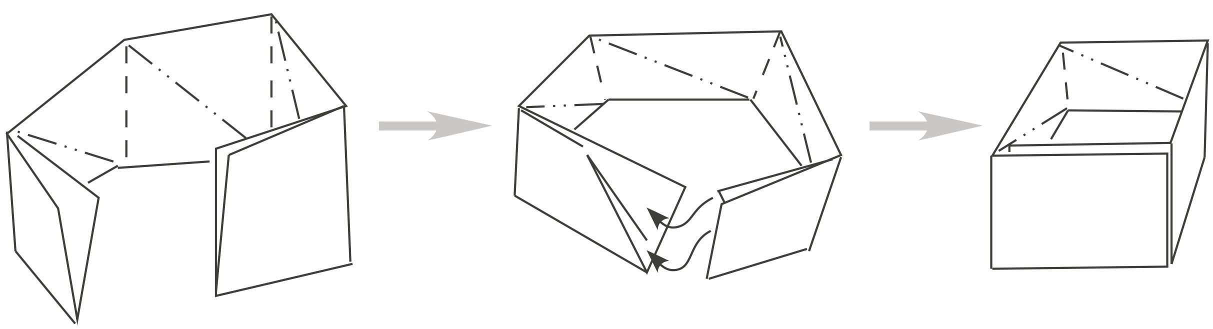
—··—··—	mountain crease
– – – – –	valley crease

   When folded, a *mountain crease* has a cross-section that looks like a $\wedge$; likewise, a *valley crease* has a cross-section that looks like a $\vee$. Now, with the crease pattern facing up, fold and unfold each diagonal crease line individually: make sure to fold only along the diagonals and do not extend the creases into the rest of the paper. It really helps if the creases are made as sharp as possible (by using the back of one's fingernails or another creasing device).

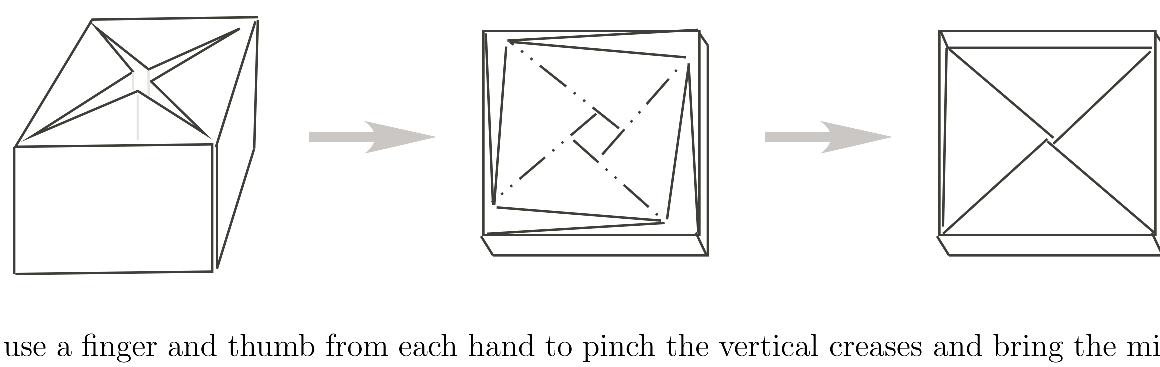
3. Mountain fold along the lower horizontal line and then the upper horizontal line, making sure to keep the diagonal creases on the outside.

4. Holding the paper so the diagonal creases are facing up, valley fold along the vertical crease lines. Fold and unfold each crease separately. Again, make these creases as sharp as possible.

5. With the creased diagonals on the inside, bring the two ends of the paper together by overlapping the first and the last segments. Interleave and tuck the flaps as shown below.



6. Now for the twist! This forms the bottom of the box. It can be challenging, so be patient with yourself. The goal is to simultaneously fold the diagonal and vertical edges of the inner layer of paper so that they twist, interleave, and lie flat. Here is one way to accomplish the goal:



   First, use a finger and thumb from each hand to pinch the vertical creases and bring the midpoints of all four top edges of the inner layer together. Then twist counterclockwise while pushing all four pinched points into the bottom corners using the valley folds along the diagonals of the segments. The edges of the paper will open like a flower. If you have difficulty, it may help to use paper clips or tape to hold the overlapped segments together.

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7. Cut out your copy of the crease pattern for the hexagon-base twist box, again only cutting along the solid lines.

   Repeat steps 2–6 on this copy. You will need to involve more of your fingers in pinching prior to the final twist.

8. Cut along the solid lines of your copy of the crease pattern for the pentagon-base twist box.

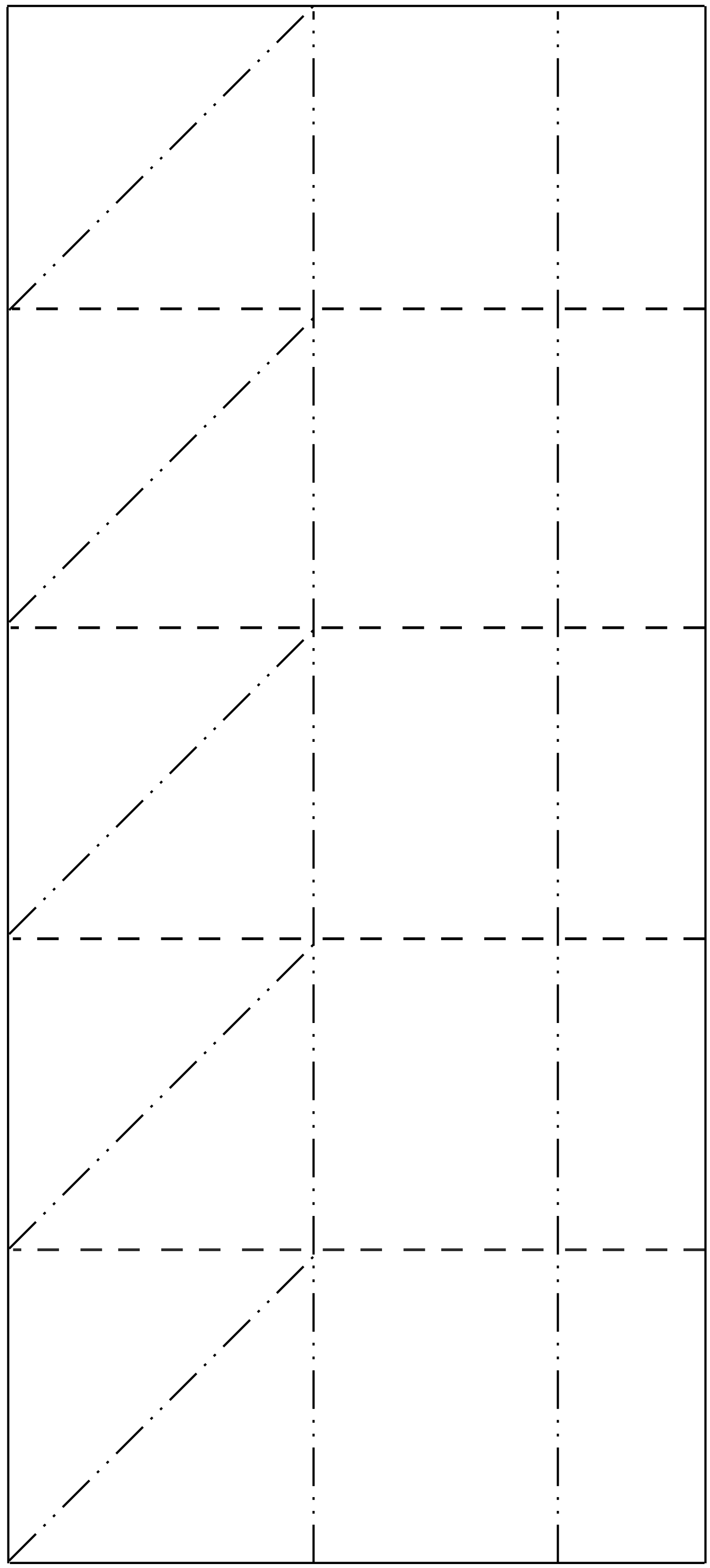
   Again repeat steps 2–6 on this copy. This time the corners of the inner layer will not land in the corners of the box, so you will have to add some creases and perhaps tuck away some extra bits of paper to get the bottom to lie flat.

9. Again, cut along the solid lines of your copy of the crease pattern for the triangle-base twist box.

   And, again repeat steps 2–6 on this copy. The corners of the inner layer will again not land in the corners of the box, so you will have to add some creases and perhaps tuck away some extra bits of paper to get the bottom to lie flat.

Congratulations! You now have four (hopefully) lovely boxes!

Crease Pattern: Square-base Twist Box



Please see the **Activity: Fold Four Boxes** handout for folding instructions.

—·· — ·· — mountain crease

— — — — valley crease

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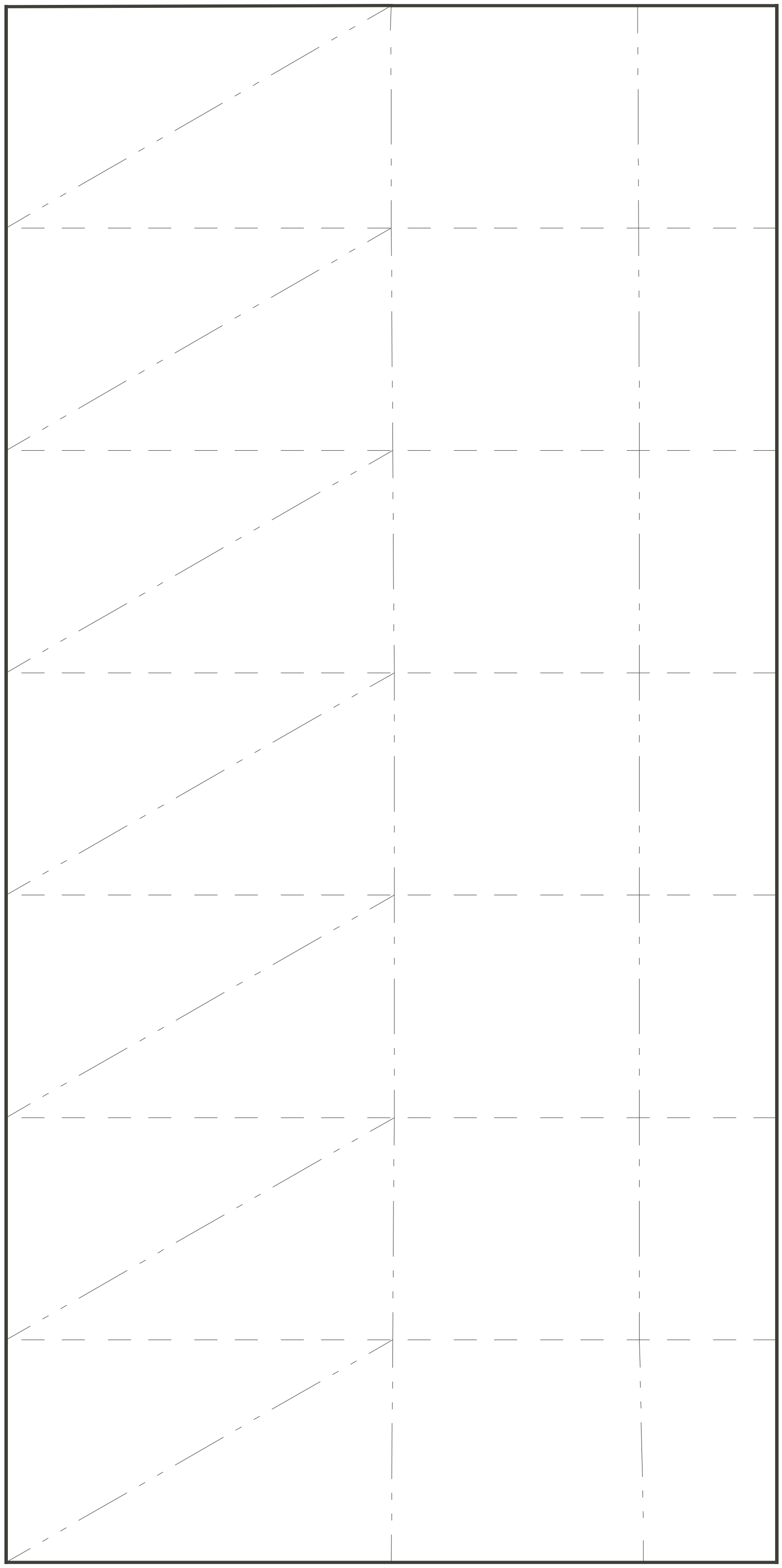
Crease Pattern: Hexagon-base Twist Box



Please see the **Activity: Fold Four Boxes** handout for folding instructions.

mountain crease

valley crease

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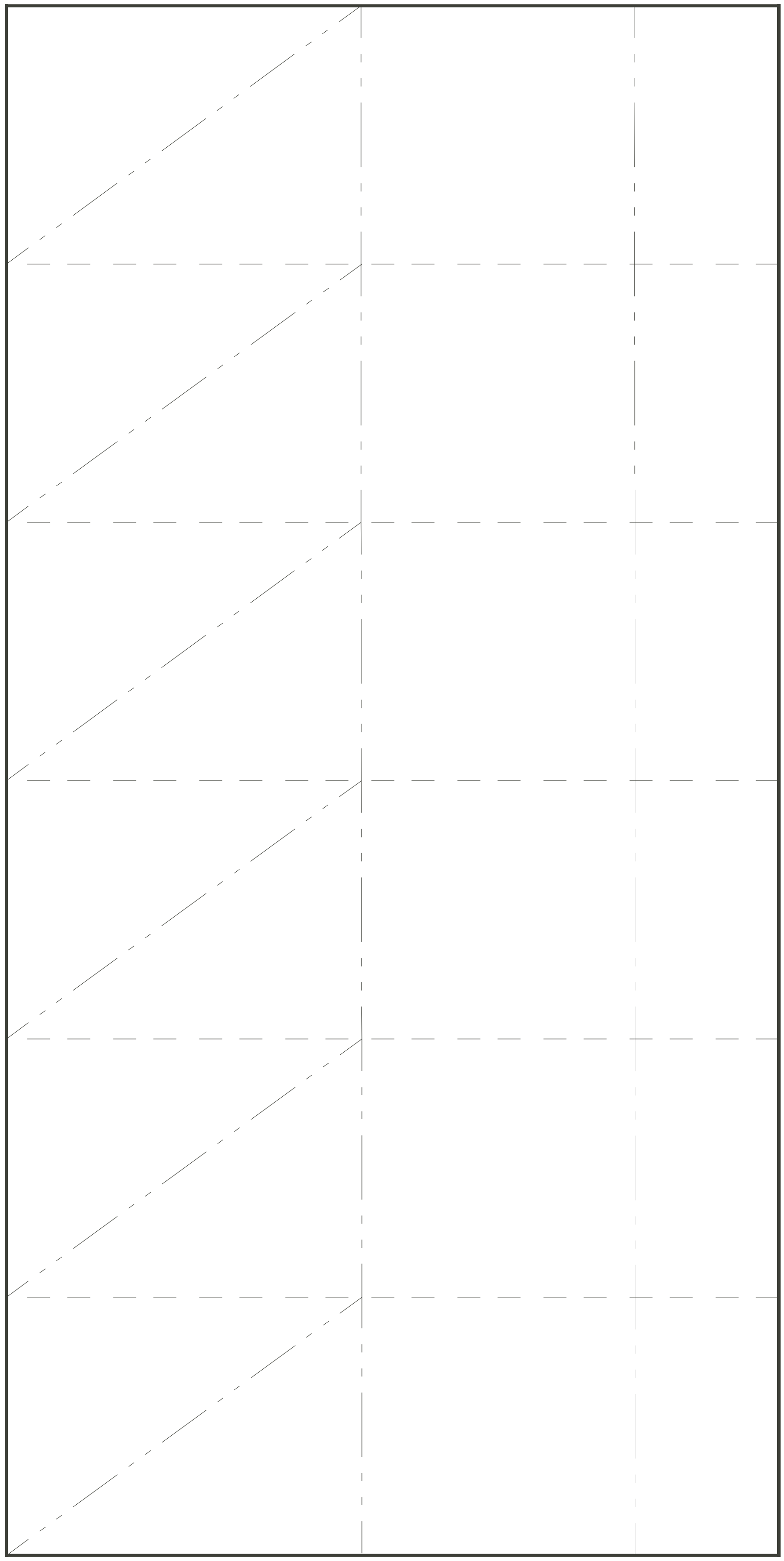
Crease Pattern: Pentagon-base Twist Box



Please see the **Activity: Fold Four Boxes** handout for folding instructions.

—·—·— mountain crease

– – – – valley crease

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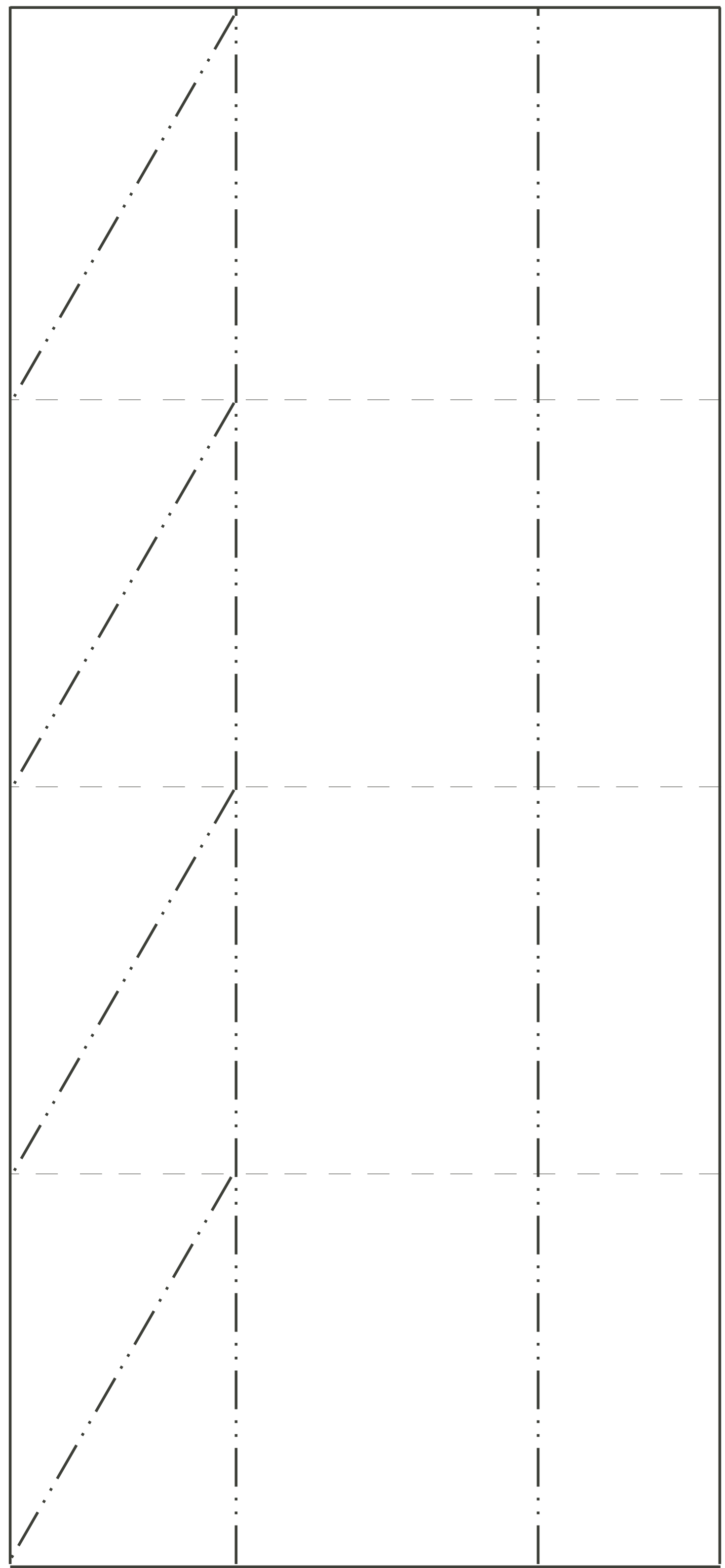
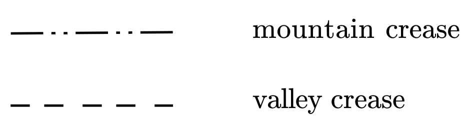
Crease Pattern: Triangle-base Twist Box



Please see the **Activity: Fold Four Boxes** handout for folding instructions.



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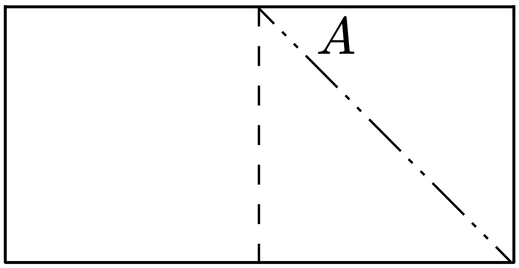
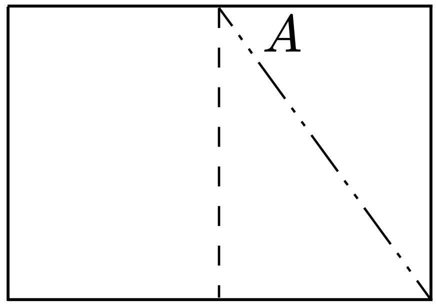
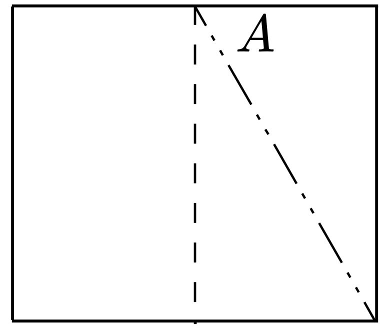
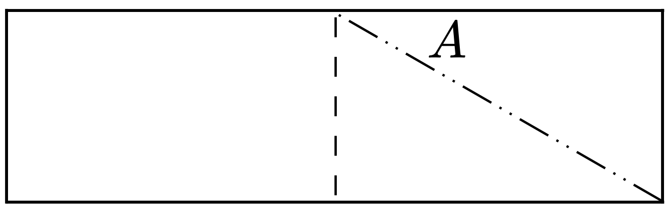
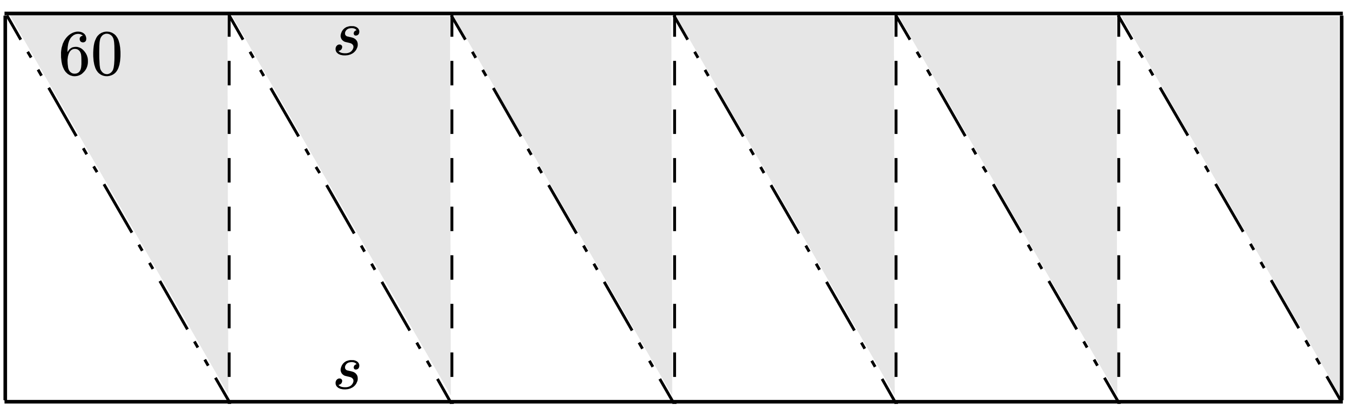
Crease Patterns for the Folding Rectangles into Regular Polygons Activity











Please see the **Activity: Folding Rectangles into Regular Polygons** handout for folding instructions.

— · · — · · — mountain crease

— — — — valley crease

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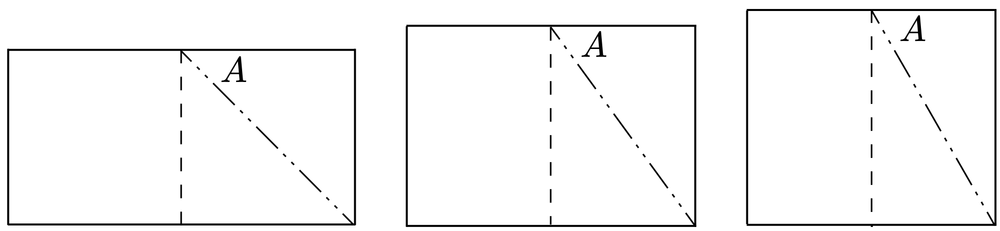
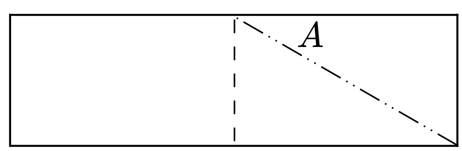
Activity: Folding Rectangles into Regular Polygons

In this activity we will examine only a portion of the crease pattern for the polygon-base boxes.

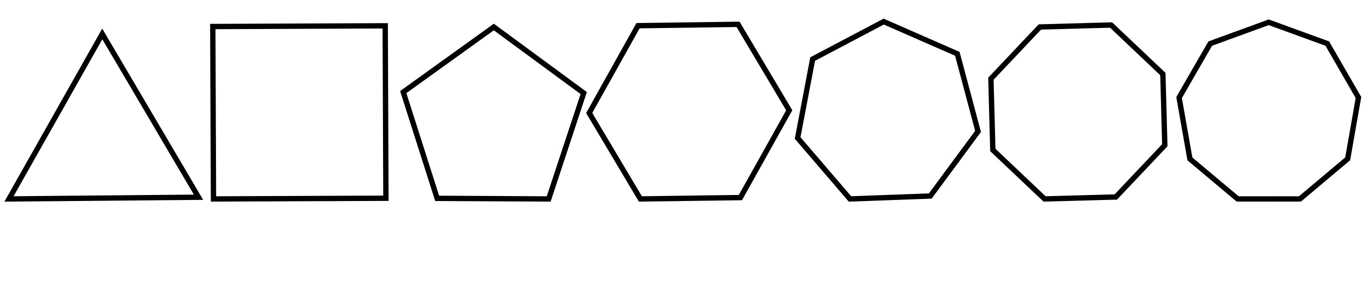
1. For each of the crease patterns below (taken from the crease patterns for the square-, pentagon- and hexagon-base boxes), measure the angle A, then cut out the copies of these crease patterns on the handout and fold them. How is the folded angle related to the angle A? Why?



2. Below is part of the crease pattern from the triangle-base box. Measure the angle A, then cut out the copy of the crease pattern on the handout and fold it. How is the folded angle related to the angle A? Why is it different in this case? For what other values of A might something similar happen?



3. Measure the interior angles of each regular polygon shown below, and find a formula for the interior angle of a regular polygon in terms of n, the number of sides of the regular polygon.



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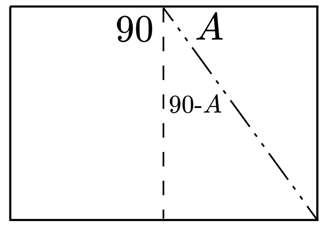
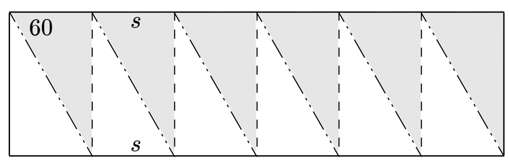
4. What would the angle A in the crease pattern below need to be to produce the interior angles of a regular heptagon (7 sides) or octagon (8 sides)?



5. Cut out the long crease pattern on the handout (pictured below in miniature) and fold it—make the folds from left to right, in order. After folding all creases, one at a time, tuck the last flap under the first flap to obtain an all-grey hexagon facing up and an all-white hexagon facing down. A regular hexagon is equilateral (has equal side lengths) AND equiangular (has equal interior angles). Use only crease-pattern information and what you learned about folding angles from Problem 1 to explain why the crease pattern should always fold the rectangles into a *regular* hexagon (even if yours isn't exactly perfect).



6. Measure s and use trigonometry and properties of triangles—don't measure—to determine the lengths of all the other edges of the triangles and rectangles in the crease pattern from Problem 5. Give the lengths in terms of s if possible.

7. Get onto GeoGebra (free at www.geogebra.org) or Geometers Sketchpad and make a crease pattern of rectangles to fold into an octagon based on your understanding above. Try folding it; does it work?

Activity: Find crease patterns for n-gon-base twist boxes

The crease patterns for the square-base, hexagon-base, pentagon-base, and triangle-base boxes (that hopefully you folded from **Activity: Fold Four Boxes**!) are structurally similar:

- There are three rows of rectangles, the top one of which has diagonal mountain creases.
- The rectangles are separated vertically by valley creases.
- The rectangles are separated horizontally by mountain creases.

We will use these similarities to understand the general construction of crease patterns for n-gon-base twist boxes. Refer to copies of the crease patterns from the **Crease Pattern: Square-base Twist Box**, **Crease Pattern: Hexagon-base Twist Box**, **Crease Pattern: Pentagon-base Twist Box**, and **Crease Pattern: Triangle-base Twist Box** worksheets for all of the questions below.

1. Which parts of the crease pattern form the polygonal base of the box? Which parts form the sides of the box?

2. Count the number of rectangles in each row of the crease pattern. How does this relate to the number of sides of the base of the box? How many rectangles-per-row should there be for an octagon-base twist box? . . . for an n-gon-base twist box?

3. In the four crease patterns on the worksheets, the third-row rectangles are shorter than the second-row rectangles. Why do none of the crease patterns have third-row rectangles that are taller than the second-row rectangles?

4. Choose one of your folded boxes (but, for ease of un- and re-folding, not the triangle-base one) and shade the inside of the base. Unfold the box to see what portions of the paper correspond to the base. (Or you can fold a second crease pattern for the same box.) Now refold just enough creases to form one vertex of the base polygon. Which angles in the crease pattern form the interior angle of the polygon? Using this information, determine what these angles should be on the crease pattern for an octagon-base twist box. Can you generalize to an n-gon-base twist box?

5. Get onto GeoGebra (free at `www.geogebra.org`) or Geometer's Sketchpad and make a crease pattern for an octagon-base twist box, based on the understanding you developed above. Try folding it; does it work?

Activity: Analyze "clean" vs. "messy" box-bottom twists

As you surely noticed while constructing the four boxes from **Activity: Fold Four Boxes**, sometimes the corners of the interior twist line up with the external corners of the box, and sometimes they do not. Here we will discover the pattern and mathematics behind this phenomenon.

1. Which of the four boxes have "clean" bottoms (where the corners of the interior twist line up with the corners of the box) and which have "messy" bottoms (where the corners of the interior twist do not line up with the corners of the box)? Make a conjecture about which n-gon-base twist boxes have "clean" bottoms and which have "messy" bottoms.

2. In a "messy" box, where *do* the corners of the interior twist seem to end up?

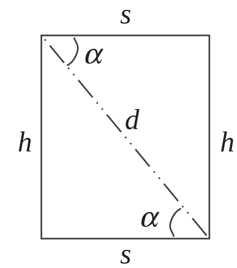
3. Choose one of your "messy" folded boxes and trace the base on a separate piece of paper. Unfold the box to step 4 of the folding instructions. Hold an end segment of the strip perpendicular to a side of the traced polygon. (The top of a rectangle with diagonal creases should line up with one of the sides of the polygon.) Now separate the inner layer (with the diagonal creases) and use it to form part of the base of the box. Try to add a second segment of the strip to this partial box formation. What does this process tell you about why the bottom of the completed box is "messy"?

4. Choose one of your "clean" folded boxes and trace the base on a separate piece of paper. Unfold the box to step 4 of the folding instructions. Hold an end segment of the of the strip perpendicular to a side of the traced polygon. Now separate the inner layer (with the diagonal creases) and use it to form part of the base of the box. Try to add a second segment of the strip to this partial box formation. How does this situation differ from the "messy" case?

5. Consider the properties that n-gons have for different values of n. In general, what property of n-gons determines which n-gon-base twist boxes have "clean" bottoms and which have "messy" bottoms, and why?

Activity: What's up with h as n gets larger and larger?

For this activity, we will focus on the triangle that is repeated across the top of an n-gon-base twist box crease pattern.



To the right, we have two copies of the basic triangle that together form a rectangle of the crease pattern. We have labeled the triangle base as s (because it forms a side length of the regular-polygon base of the box), the height of the the triangle as h, the length of the diagonal as d, and one angle as α. The labeled angle, α, is half of the interior angle of the base n-gon, or $\alpha = \dfrac{\pi(n-2)}{2n}$.

1. Find a formula for h in terms of s and α.

2. Now, to make the results for different polygon-base twist boxes uniform, we will let the perimeter of the polygon be 1. What is s in terms of n?

3. Convert your above formula for h to be in terms of n instead of s and α.

4. To construct a different algebraic formula for h, find a formula for n in terms of α. Use this to obtain a formula for h solely in terms of α.

5. Now let's do some computations. Fill in this table of values of n vs. values of h:

n	3	4	5	6	7	8	9	10	12	16	20	50	100	200	500	1000
h																

6. Just for fun, compute $\lim_{n\to\infty} \alpha$.

7. Make a conjecture about $\lim_{n\to\infty} h$, and compute that limit to verify that your conjecture is correct…or to be surprised by a relationship between h and π. (Not all formulas are equally good for the purposes of this calculation.)

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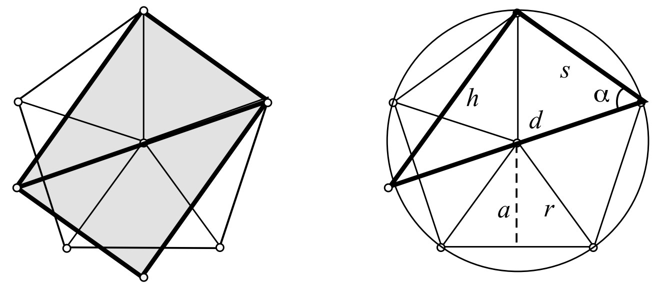
Activity: How much excess *is* there in a messy box bottom?

Looking inside the four boxes you folded, you may be quick to conjecture that the annoying not-fitting interior twist happens when n is odd (and you would be correct).

To the right is shown one rectangle of the interior twist for a pentagon-base twist box. This shows the extent to which the twist paper juts out past the base. The pentagon stands in for a generic n-gon for n odd. We show the n-gon partitioned into n isosceles triangles; a is an altitude of any of these isosceles triangles, and r is the radius of the circle that circumscribes the n-gon.



It is safe to assume that both endpoints of d lie on this circle. (Can you explain why?)

1. Mark on the diagram the length x by which the interior twist exceeds the polygonal base.

2. This length, x, is $d -$ *something*, but how can we measure *something*? Use the diagram together with geometry and/or trigonometry to determine *something* in terms of known quantities (i.e. s, α, d, h).

3. Consider how x changes as n changes and find a formula relating the two variables.

4. Compute $\lim_{n\to\infty} x$ and explain what this means in practice for folding n-gon-base twist boxes.

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